

EIGENVALUE ESTIMATES FOR MAGNETIC SCHRÖDINGER OPERATORS IN DOMAINS

RUPERT L. FRANK, ARI LAPTEV, AND STANISLAV MOLCHANOV

ABSTRACT. Inequalities are derived for sums and quotients of eigenvalues of magnetic Schrödinger operators with non-negative electric potentials in domains. The bounds reflect the correct order of growth in the semi-classical limit.

1. INTRODUCTION AND MAIN RESULTS

In this paper we derive inequalities for the eigenvalues of magnetic Schrödinger operators in a domain. Such inequalities have received a lot of attention in the case where both the electric and the magnetic potential are absent. If $0 < \lambda_1^0 < \lambda_2^0 \leq \dots$ denote the eigenvalues of the Dirichlet Laplacian in a domain $\Omega \subset \mathbb{R}^d$ of finite measure, then the Berezin-Li-Yau inequality [B, LY] states that

$$(1.1) \quad \sum_j (\lambda - \lambda_j^0)_+ \leq \frac{2}{d+2} v_d |\Omega| \lambda^{1+d/2}$$

for any $\lambda > 0$. Here

$$(1.2) \quad v_d := |\mathbb{S}^{d-1}|/d = \pi^{d/2}/\Gamma(1+d/2)$$

denotes the volume of the d -dimensional unit ball. Performing a Legendre transform in (1.1) one obtains (see [LW])

$$(1.3) \quad \sum_{j=1}^k \lambda_j^0 \geq \frac{4\pi^2 d}{d+2} v_d^{-2/d} |\Omega|^{-2/d} k^{1+2/d}.$$

The important point of the estimates (1.1) and (1.3) is that they are sharp in the semi-classical limit. Indeed, according to Weyl's theorem, see, e.g., [LL, RS],

$$(1.4) \quad \lambda_k^0 \sim 4\pi^2 v_d^{-2/d} |\Omega|^{-2/d} k^{2/d}, \quad k \rightarrow \infty,$$

which implies sharpness of (1.1) as $\lambda \rightarrow \infty$ and of (1.3) as $k \rightarrow \infty$.

Given the *lower* bounds (1.1) and (1.3) on eigenvalue sums it is natural to ask for *upper* bounds which have the correct asymptotic behavior. In [L]

it is shown that

$$(1.5) \quad \sum_j (\lambda - \lambda_j^0)_+ \geq \frac{2}{(d+2)(2\pi)^d} v_d \|\omega\|_\infty^{-2} (\lambda - \lambda_1^0)_+^{1+d/2}$$

for all $\lambda > 0$ where $\omega \geq 0$ denotes the normalized eigenfunction corresponding to the eigenvalue λ_1^0 . Later it was observed in [H] that by an isoperimetric inequality of Chiti [C] $\|\omega\|_\infty$ can be bounded from above in terms of the eigenvalue λ_1^0 . This yields the estimate

$$(1.6) \quad \sum_j (\lambda - \lambda_j^0)_+ \geq \frac{2}{d+2} H_d^{-1} (\lambda_1^0)^{-d/2} (\lambda - \lambda_1^0)_+^{1+d/2}$$

with a non-sharp but explicit constant

$$(1.7) \quad H_d := \frac{2d}{j_{(d-2)/2}^2 J_{d/2}^2(j_{(d-2)/2})}.$$

Here J_ν denotes the Bessel function of order ν and j_ν its first positive zero. Passing to the Legendre transform one finds

$$(1.8) \quad \sum_{j=1}^k (\lambda_j^0 - \lambda_1^0) \leq \frac{d}{d+2} H_d^{2/d} \lambda_1^0 k^{1+2/d}.$$

Both inequalities (1.6) and (1.8) reflect the correct order of growth with respect to k as given by (1.4). However, note that λ_1^0 in these inequalities replaces $|\Omega|^{-2/d}$ in (1.1) and (1.3). This can be used to derive upper bounds on the eigenvalue ratios λ_k/λ_1 from (1.6) and (1.8). The sharp constant in the case $k = 2$ was found in [AB3].

The purpose of the present paper is to show that the inequality (1.6) and its corollaries extends, with the same constant, to the case of a non-trivial magnetic field and a non-negative electric potential. More precisely, let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a domain of finite measure, let $V \in L_{1,loc}(\Omega)$ be non-negative and let $A \in L_{2,loc}(\Omega, \mathbb{R}^d)$ be arbitrary. We define the Schrödinger operator

$$H := (D - A)^2 + V \quad \text{in } L_2(\Omega)$$

by closing its quadratic form on $C_0^\infty(\Omega)$. We assume that this operator has compact resolvent and denote its eigenvalues by $0 < \lambda_1 \leq \lambda_2 \leq \dots$, where each eigenvalue is repeated according to multiplicities.

Our main result is

Theorem 1.1. *For any $\lambda \geq 0$ one has*

$$(1.9) \quad \sum_j (\lambda - \lambda_j)_+ \geq \frac{2}{d+2} H_d^{-1} \lambda_1^{-d/2} (\lambda - \lambda_1)_+^{1+d/2}$$

with the constant H_d from (1.7). Moreover, for any $k \in \mathbb{N}$ one has

$$(1.10) \quad \sum_{j=1}^k (\lambda_j - \lambda_1) \leq \frac{d}{d+2} H_d^{2/d} \lambda_1 k^{1+2/d}.$$

From these estimates on sums of eigenvalues one easily obtains estimates on the ratio λ_{k+1}/λ_1 .

Corollary 1.2. *For any $k \in \mathbb{N}$ one has*

$$(1.11) \quad \lambda_{k+1} \leq \left(1 + \left(1 + \frac{d}{2}\right)^{2/d} H_d^{2/d} k^{2/d}\right) \lambda_1,$$

$$(1.12) \quad \lambda_{k+1} \leq \left(1 + \frac{4}{d}\right) \left(1 + \frac{d}{d+2} H_d^{2/d} k^{2/d}\right) \lambda_1.$$

We emphasize that both estimates in Corollary 1.2 have the correct order of growth as $k \rightarrow \infty$. Indeed, the asymptotics (1.4) are also valid for the eigenvalues λ_k provided V , A and Ω are sufficiently regular.

For large values of k the bound (1.12) is better than (1.11). For small values of k (1.11) is better than (1.12), but an even better bound is given by

$$(1.13) \quad \lambda_{k+1} \leq \left(1 + \frac{4}{d}\right)^k \lambda_1, \quad k \in \mathbb{N}.$$

The latter is a Payne-Pólya-Weinberger-type bound. The different constants in the bounds (1.11), (1.12) and (1.13) are further discussed in [H].

We emphasize that all our bounds are known in the case $V \equiv A \equiv 0$, see [H]. Our point is that they extend, with the same constants, to the case of a non-trivial magnetic field and a non-negative electric potential. This is non-trivial since the diamagnetic inequality apparently gives only information about the lowest eigenvalue. In particular, it seems not clear to us whether the PPW-conjecture (see [AB3]), namely that the quotient λ_2/λ_1 is maximized for a circle and $V \equiv 0$, extends to the magnetic case.

We remark that our inequalities are also true if $d = 1$, but in this case much stronger results are known. (Recall that any magnetic field in one dimension can be gauged away.) In particular, if $V \geq 0$, then one has the sharp inequality $\lambda_k/\lambda_1 \leq k^2$, see [AB2]. A beautiful proof of this inequality for $k = 2$ is given in [AB1].

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2. PROOF OF THEOREM 1.1

2.1. Main ingredients. Our proof of Theorem 1.1 is based on three propositions which might be of independent interest. We defer their proofs to the following sections. First, we derive a lower bound on the eigenvalue sum $\sum_j (\lambda - \lambda_j)_+$ in terms of λ and the maximum of the lowest eigenfunction.

Proposition 2.1. *Let ω be an eigenfunction corresponding to the lowest eigenvalue λ_1 of H with $\|\omega\| = 1$. Then for any $\lambda \geq 0$ one has*

$$(2.1) \quad \sum_j (\lambda - \lambda_j)_+ \geq \frac{2v_d}{(d+2)(2\pi)^d} \|\omega\|_\infty^{-2} (\lambda - \lambda_1)_+^{1+d/2}$$

with v_d given in (1.2).

In the case $V \equiv A \equiv 0$ this is Theorem 4.1 in [L]. It turned out that almost the same proof holds even for arbitrary A and $V \geq 0$. We emphasize that in the magnetic case the lowest eigenvalue λ_1 may be degenerate and $\|\omega\|_\infty$ may depend on the choice of the eigenfunction. Some inequalities of the same spirit have been obtained in [S] for elliptic differential operators with variable coefficients.

In order to use Proposition 2.1 for the proof of Theorem 1.1 we need to estimate the maximum of the lowest eigenfunction in terms of the corresponding eigenvalue.

Proposition 2.2. *Let λ be an eigenvalue of H and ω a corresponding eigenfunction. Then for any $p > 0$*

$$(2.2) \quad \|\omega\|_\infty \leq C_d(p) \lambda^{d/2p} \|\omega\|_p,$$

with the universal constant

$$C_d(p) = \left(2^{p(d-2)/2} \Gamma(d/2)^p |\mathbb{S}^{d-1}| \int_0^{j_{(d-2)/2}} J_{(d-2)/2}^p(r) r^{-p(d-2)/2+d-1} dr \right)^{-1/p}.$$

In particular,

$$(2.3) \quad C_d(2) = \left(\pi^{d/2} 2^{d-2} \Gamma(d/2) j_{(d-2)/2}^2 J_{d/2}^2(j_{(d-2)/2}) \right)^{-1/2}.$$

In the case $V \equiv A \equiv 0$ this is a result of Chiti [C]. Non-negative electric potentials can be included without major changes, as was done in [AB3]. Using Kato's inequality we shall show that the same proof even works for a non-trivial magnetic field.

We emphasize that (2.2) is an isoperimetric result: If Ω is a ball, $V \equiv \text{curl } A \equiv 0$ and λ is the lowest eigenvalue of H , then one has equality in (2.2). This explains the value of the constant $C_d(p)$. To discuss the cases of equality would be beyond the scope of this paper.

The third ingredient employed in the proof of our main results is the following Yang-type inequality.

Proposition 2.3. *For any $k \in \mathbb{N}$ one has*

$$(2.4) \quad \sum_{j=1}^k (\lambda_{k+1} - \lambda_j) \left(\lambda_{k+1} - \left(1 + \frac{4}{d} \right) \lambda_j \right) \leq 0.$$

This result in the case $V \equiv A \equiv 0$ is due to Yang [Y]. A slightly stronger result in the case of arbitrary V and A is contained in Theorem 5 in [HS]; for results in the same spirit see [LP]. A different proof (for arbitrary $V \geq 0$,

but $A \equiv 0$) was given in [A]. In Section 5 we shall show that this proof extends to the case of a non-trivial magnetic field.

One can deduce from Theorem 2.3 several weaker but more explicit eigenvalue estimates.

Corollary 2.4. *For any $k \in \mathbb{N}$ one has*

$$\begin{aligned} \text{(Y)} \quad & \lambda_{k+1} \leq \left(1 + \frac{4}{d}\right) \frac{1}{k} \sum_{j=1}^k \lambda_j, \\ \text{(HP)} \quad & \frac{1}{k} \sum_{j=1}^k \frac{\lambda_j}{\lambda_{k+1} - \lambda_j} \geq \frac{d}{4}, \\ \text{(PPW)} \quad & \lambda_{k+1} - \lambda_k \leq \frac{4}{d} \frac{1}{k} \sum_{j=1}^k \lambda_j. \end{aligned}$$

In the case $V \equiv A \equiv 0$ these estimates are known as second Yang inequality, Hile-Protter inequality and Payne-Pólya-Weinberger inequality, respectively. For any k one has the implications

$$(2.5) \quad (2.4) \Rightarrow \text{(Y)} \Rightarrow \text{(HP)} \Rightarrow \text{(PPW)}.$$

The proof of (2.5) is based on general arguments (see [A]) and will not be reproduced here.

Remark 2.5. Note that (PPW) implies (1.13). Indeed, estimating $\frac{1}{k} \sum_{j=1}^k \lambda_j \leq \lambda_k$ we obtain $\lambda_{k+1}/\lambda_k \leq 1 + 4/d$. Multiplying these estimates for consecutive values of k we obtain (1.13).

2.2. Proof of the main results. We now show how Theorem 1.1 and Corollary 1.2 follow from Propositions 2.1, 2.2 and 2.3.

Proof of Theorem 1.1. To get (1.9) we simply combine the estimates (2.1) and (2.2) and note that $H_d = (2\pi)^d v_d^{-1} C_d(2)^2$.

To deduce (1.10) from (1.9) we use the Legendre transform argument from [LW]. Indeed, for a function h on $[0, \infty)$ put $Lh(p) := \sup_{\lambda \geq 0} (p\lambda - h(\lambda))$. So if $f(\lambda) := \sum_j (\lambda - \lambda_j)_+$ and $g(\lambda) := \frac{2}{d+2} H_d^{-1} \lambda_1^{-d/2} (\lambda_1 - \lambda)_-^{1+d/2}$, then

$$Lf(p) = \{p\} \lambda_{[p]+1} + \sum_{j=1}^{[p]} \lambda_j$$

(with $\{p\}$, $[p]$ the fractional and integer parts of p) and

$$Lg(p) = p\lambda_1 + \frac{d}{d+2} H_d^{2/d} \lambda_1 p^{1+2/d}.$$

Now $f \geq g$ implies $Lf \leq Lg$, and evaluating the latter at an integer $p = k \in \mathbb{N}$ we obtain (1.10). \square

Proof of Corollary 1.2. For the proof of (1.11) we put $\lambda = \lambda_{k+1}$ in (1.9) and estimate $\sum_{j=1}^k (\lambda_{k+1} - \lambda_j) \leq k(\lambda_{k+1} - \lambda_1)$. For the proof of (1.11) we combine (Y) with (1.10). \square

3. UPPER BOUND ON EIGENVALUE SUMS

In this section we prove Proposition 2.1. We let $u_1 := \omega$ and denote by u_j , $j \in \mathbb{N}$, orthonormal eigenfunctions corresponding to the eigenvalues λ_j . We put $\theta_\xi(x) := e^{i\xi \cdot x} \omega(x)$, $\tilde{\omega} := \|\omega\|_\infty$ and calculate as in [L] using Plancherel's equality

$$\begin{aligned}
 (3.1) \quad \sum_j (\lambda - \lambda_j)_+ &\geq \tilde{\omega}^{-2} \sum_j (\lambda - \lambda_j)_+ \int_\Omega |\bar{\omega} u_j|^2 dx \\
 &= (2\pi)^{-d} \tilde{\omega}^{-2} \sum_j (\lambda - \lambda_j)_+ \int_{\mathbb{R}^d} \left| \int_\Omega \bar{\omega}(x) u_j(x) e^{-i\xi \cdot x} dx \right|^2 d\xi \\
 &= (2\pi)^{-d} \tilde{\omega}^{-2} \int_{\mathbb{R}^d} \sum_j (\lambda - \lambda_j)_+ |(\theta_\xi, u_j)|^2 d\xi \\
 &= (2\pi)^{-d} \tilde{\omega}^{-2} \int_{\mathbb{R}^d} \left(\int_0^\infty (\lambda - \nu)_+ d(E(\nu) \theta_\xi, \theta_\xi) \right) d\xi.
 \end{aligned}$$

Here $dE(\nu)$ denotes the spectral measure of H . We note that

$$\int_0^\infty d(E(\nu) \theta_\xi, \theta_\xi) = \|\theta_\xi\|^2 = \|\omega\|^2 = 1.$$

Since the function $\nu \mapsto (\lambda - \nu)_+$ is convex, we can apply Jensen's inequality to get

$$\begin{aligned}
 (3.2) \quad \int_0^\infty (\lambda - \nu)_+ d(E(\nu) \theta_\xi, \theta_\xi) &\geq \left(\lambda - \int_0^\infty \nu d(E(\nu) \theta_\xi, \theta_\xi) \right)_+ \\
 &= \left(\lambda - \int_\Omega (|(D - A)\theta_\xi|^2 + V|\theta_\xi|^2) dx \right)_+.
 \end{aligned}$$

We shall need the following simple

Lemma 3.1. *For any $\xi \in \mathbb{R}^d$ one has*

$$(3.3) \quad \int_\Omega (|(D - A)\theta_\xi|^2 + V|\theta_\xi|^2) dx = \lambda_1 + |\xi|^2.$$

Proof. We denote by h the quadratic form of the operator H , i.e.,

$$h[u] = \int_\Omega (|(D - A)u|^2 + V|u|^2) dx.$$

An easy calculation shows that

$$(3.4) \quad h[\theta_\xi] = \lambda_1 + |\xi|^2 + 2j \cdot \xi$$

where

$$j := \operatorname{Im} \int_\Omega \bar{\omega} (\nabla - iA) \omega dx,$$

and we have to show that $j = 0$. Assume to the contrary that $j \neq 0$ and choose ξ_0 such that $j \cdot \xi_0 < 0$. Then in view of (3.4), $h[\theta_{\varepsilon\xi_0}] < \lambda_1$ for all sufficiently small $\varepsilon > 0$. But this contradicts the variational characterization of the lowest eigenvalue. \square

Combining (3.1), (3.2) and (3.3) we arrive at

$$\begin{aligned} \sum_j (\lambda - \lambda_j)_+ &\geq (2\pi)^{-d} \tilde{\omega}^{-2} \int_{\mathbb{R}^d} (\lambda - \lambda_1 - |\xi|^2)_+ d\xi \\ &= (2\pi)^{-d} \tilde{\omega}^{-2} v_d \frac{2}{d+2} (\lambda - \lambda_1)_+^{1+d/2}, \end{aligned}$$

which proves the assertion.

4. EIGENFUNCTION ESTIMATES FOR MAGNETIC SCHRÖDINGER OPERATORS

4.1. A non-sharp bound. As we pointed out after Proposition 2.2, the constants in (2.2) are sharp. Before giving its rather involved proof we would like to derive a (non-sharp) estimate which still has the correct form needed in the proof of Theorem 1.1. More precisely, we shall prove that if ω is an eigenfunction corresponding to an eigenvalue λ of H , then

$$\|\omega\|_\infty \leq \tilde{C}_d \lambda^{d/4} \|\omega\|_2,$$

with

$$\tilde{C}_d = (e/d\pi)^{d/4}.$$

To see this, we note that by the diamagnetic inequality and the Feynman-Kac formula [D] one has for all $t > 0$ and all $x \in \Omega$ the pointwise estimate

$$\left| e^{-t\lambda} \omega(x) \right| = \left| (e^{-tH} \omega)(x) \right| \leq (e^{t\Delta} |\omega|)(x).$$

Here $|\omega|$ is extended by 0 to \mathbb{R}^d and $e^{t\Delta}$ denotes the heat semi-group on \mathbb{R}^d . Using the explicit expression of its kernel we can estimate

$$(e^{t\Delta} |\omega|)(x) \leq (4\pi t)^{-d/2} \left(\int_{\mathbb{R}^d} e^{-|x-y|^2/2t} dy \right)^{1/2} \|\omega\|_2 = (4\pi t)^{-d/4} \|\omega\|_2.$$

Combining the previous two relations and optimizing with respect to t we arrive at

$$|\omega(x)| \leq \inf_{t>0} e^{t\lambda} (4\pi t)^{-d/4} \|\omega\|_2 = (e/d\pi)^{d/4} \lambda^{d/4} \|\omega\|_2,$$

as claimed.

4.2. Proof of Proposition 2.2. Now we turn to the sharp bound (2.2). We introduce the ball

$$S := \{x \in \mathbb{R}^d : |x| < j_{(d-2)/2} \lambda^{-1/2}\}$$

and the function

$$z(x) := |x|^{-(d-2)/2} J_{(d-2)/2}(\sqrt{\lambda}|x|).$$

Note that S is the ball centered at the origin for which the Dirichlet problem (with $V \equiv A \equiv 0$) has lowest eigenvalue λ . The function z is the corresponding eigenfunction. Next we introduce the ball

$$(4.1) \quad \Omega^* := \{x \in \mathbb{R}^d : |x| < v_d^{-1/d} |\Omega|^{1/d}\}$$

centered at the origin with the same volume as Ω . The constant v_d is the volume of the d -dimensional unit ball, see (1.2).

Lemma 4.1. *One has $S \subset \Omega^*$.*

Proof. We denote by $\lambda_1(-\Delta, \mathcal{O})$ the lowest eigenvalue of the Dirichlet Laplacian on a domain \mathcal{O} . Note that by definition of S

$$\lambda_1(-\Delta, S) = \lambda.$$

Now the variational principle together with the diamagnetic inequality and the non-negativity of V imply

$$\lambda \geq \lambda_1(-\Delta, \Omega).$$

Finally, the Faber-Krahn inequality states that

$$\lambda_1(-\Delta, \Omega) \geq \lambda_1(-\Delta, \Omega^*).$$

Combining the previous relations we obtain $\lambda_1(-\Delta, S) \geq \lambda_1(-\Delta, \Omega^*)$. By the variational principle, this implies $S \subset \Omega^*$. \square

We denote by $\omega^* : \Omega^* \rightarrow [0, \infty)$ the spherically decreasing rearrangement of ω , see [LL]. We note that z is a spherically decreasing function. (This follows either by properties of Bessel functions or by rearrangement arguments.) In particular,

$$(4.2) \quad \|z\|_\infty = z(0).$$

The core of Proposition 2.2 is the following comparison result.

Lemma 4.2. *Assume that ω is normalized such that $\|\omega\|_{L_\infty(\Omega)} = \|z\|_{L_\infty(S)}$. Then*

$$\omega^*(x) \geq z(x), \quad x \in S.$$

We defer the rather involved proof of this lemma to the next subsection. Now we shall use it to give the

Proof of Proposition 2.2. By Lemma 4.2 one has

$$\|z\|_{L_\infty(S)} \omega^*(x) \geq \|\omega\|_{L_\infty(\Omega)} z(x), \quad x \in S.$$

Hence according to rearrangement properties [LL] and Lemma 4.1

$$(4.3) \quad \|\omega\|_{L_p(\Omega)} = \|\omega^*\|_{L_p(\Omega^*)} \geq \|\omega^*\|_{L_p(S)} \geq \frac{\|z\|_{L_p(S)}}{\|z\|_{L_\infty(S)}} \|\omega\|_{L_\infty(\Omega)}.$$

The asymptotics $r^{-(d-2)/2} J_{(d-2)/2}(r) \rightarrow \Gamma(d/2)^{-1} 2^{-(d-2)/2}$ as $r \rightarrow 0$ (see [AS, (9.1.7)]) and (4.2) imply

$$\frac{\|z\|_{L_\infty(S)}}{\|z\|_{L_p(S)}} = C_d(p) \lambda^{d/2p}.$$

In the special case $p = 2$ we use (see [AS, (11.4.5)])

$$\int_0^{j_{(d-2)/2}} r J_{(d-2)/2}^2(r) dr = \frac{1}{2} j_{(d-2)/2}^2 J_{d/2}^2(j_{(d-2)/2})$$

and $|\mathbb{S}^{d-1}| = d\pi^{d/2}/\Gamma((d+2)/2)$ to get the claimed expression. \square

4.3. Proof of Lemma 4.2. Since z is a radial function, there is a function v on $[0, |S|]$ such that

$$v(v_d |x|^d) = z(x), \quad x \in S.$$

Here v_d is the volume of the unit ball, see (4.1). From the differential equation satisfied by z one derives easily the integro-differential equation

$$(4.4) \quad -v'(s) = d^{-2} v_d^{-2/d} \lambda s^{-2+2/d} \int_0^s v(t) dt, \quad 0 < s < |S|,$$

for v . Our next goal is to derive a similar integro-differential *inequality* for ω^* . Namely, we let u be the function on $[0, |\Omega|]$ such that

$$u(v_d |x|^d) = \omega^*(x), \quad x \in \Omega^*.$$

Note that u is a decreasing function with $u(0) = \|\omega\|_\infty$ and $u(|\Omega|) = 0$.

Lemma 4.3. *One has*

$$(4.5) \quad -u'(s) \leq d^{-2} v_d^{-2/d} \lambda s^{-2+2/d} \int_0^s u(t) dt, \quad 0 < s < |\Omega|.$$

Proof. We recall Kato's inequality [S1, S2],

$$\operatorname{Re}((\operatorname{sgn} w)(D - A)^2 w) \geq -\Delta|w|,$$

where $\operatorname{sgn} w(x) = \overline{w(x)}/|w(x)|$ if $w(x) \neq 0$ and $\operatorname{sgn} w(x) = 0$ otherwise. Since $V \geq 0$ we have

$$\lambda|\omega| = \operatorname{Re}((\operatorname{sgn} \omega)((D - A)^2 + V)\omega) \geq -\Delta|\omega|.$$

We integrate this inequality over the set $\{|\omega| > t\}$ and use Gauss's theorem in the form

$$-\int_{\{|w|>t\}} \Delta|w| dx = \int_{\{|w|=t\}} |\nabla|w|| d\sigma.$$

Here $d\sigma$ denotes $(d-1)$ -dimensional surface measure. We obtain

$$(4.6) \quad \lambda \int_{\{|\omega|>t\}} |\omega| dx \geq \int_{\{|\omega|=t\}} |\nabla|\omega|| d\sigma.$$

The next step is to estimate the surface integral on the RHS from below. We introduce the distribution function

$$\mu(t) := |\{x \in \Omega : |\omega| > t\}|$$

and recall the coarea formula

$$-\mu'(t) = \int_{\{|\omega|=t\}} |\nabla|\omega||^{-1} d\sigma.$$

Hence by Cauchy-Schwarz

$$\begin{aligned} \sigma(\{|\omega| = t\})^2 &\leq \left(\int_{\{|\omega|=t\}} |\nabla|\omega||^{-1} d\sigma \right) \left(\int_{\{|\omega|=t\}} |\nabla|\omega|| d\sigma \right) \\ &= -\mu'(t) \int_{\{|\omega|=t\}} |\nabla|\omega|| d\sigma. \end{aligned}$$

Combining this with the isoperimetric inequality

$$\sigma(\{|\omega| = t\}) \geq dv_d^{1/d} \mu(t)^{1-1/d}$$

we finally arrive at

$$\int_{\{|\omega|=t\}} |\nabla|\omega|| d\sigma \geq d^2 v_d^{2/d} \mu(t)^{2-2/d} (-\mu'(t))^{-1}.$$

Combining this with (4.6) we have shown that

$$\mu(t)^{2-2/d} (-\mu'(t))^{-1} \leq d^{-2} v_d^{-2/d} \lambda \int_{\{|\omega|>t\}} |\omega| dx.$$

It remains to substitute $t = u(s)$ and to note that when $u(s) \neq 0$,

$$\mu(t)^{2-2/d} (-\mu'(t))^{-1} = s^{2-2/d} (-u'(s)), \quad \int_{\{|\omega|>t\}} |\omega| dx = \int_0^s u(r) dr.$$

This proves the assertion. \square

Now everything is in place for the

Proof of Lemma 4.2. The strategy is to construct a trial function g for $-\Delta$ on S which is as good as z and hence must coincide with z . We distinguish two cases, namely whether $|S| < |\text{supp } \omega|$ or not. We begin by assuming that this strict inequality holds. Recall the definition of the functions u , v before Lemma 4.3. Both functions are decreasing and the normalization condition $\|\omega\|_{L^\infty(\Omega)} = \|z\|_{L^\infty(S)}$ means $u(0) = v(0)$. Moreover, u is strictly positive on $[0, |S|]$. (This is where we use that $|S| < |\text{supp } \omega|$.) Hence

$$c := \max_{0 \leq s \leq |S|} v(s)/u(s)$$

is finite and greater or equal to 1. We have to prove that $c = 1$. For this we choose a point $s_0 \in [0, |S|)$ such that $v(s_0) = cu(s_0)$. (The point can be chosen smaller than $|S|$ since $v(|S|) = 0$.) Defining

$$w(s) := \begin{cases} cu(s), & 0 \leq s \leq s_0, \\ v(s), & s_0 \leq s \leq |S|, \end{cases}$$

it follows from (4.4), (4.5) and the definition of c that w satisfies

$$(4.7) \quad -w'(s) \leq d^{-2}v_d^{-2/d}\lambda s^{-2+2/d} \int_0^s w(t) dt, \quad 0 < s < |S|.$$

Now we define $g(x) := w(v_d|x|^d)$ for $x \in S$ and note that $g \in \overset{\circ}{H}^1(S)$ with

$$\int_S |g|^2 dx = \int_0^{|S|} |w|^2 ds$$

and, using (4.7),

$$\begin{aligned} \int_S |\nabla g|^2 dx &= d^2 v_d^{2/d} \int_0^{|S|} |w'|^2 s^{2-2/d} ds \\ &\leq \lambda \int_0^{|S|} (-w'(s)) \int_0^s w(r) dr ds \\ &= \lambda \int_0^{|S|} |w|^2 ds. \end{aligned}$$

Recall that λ is the lowest eigenvalue of $-\Delta$ on S . Since it is simple and has z as a corresponding eigenfunction, we conclude that g is a scalar multiple of z . Hence w is a scalar multiple of v . Since w and v coincide on $[s_0, |S|]$ they coincide everywhere. Hence $cu(s) = v(s)$ for all $0 \leq s \leq s_0$. Evaluating at $s = 0$ we find $c = 1$, which proves the assertion in the case under consideration.

Now we turn to the case $|S| \geq |\text{supp } \omega|$. Note that by Lemma 4.1 we know that $S \subset \Omega^*$. We extend ω^* by 0 to S (if $|\text{supp } \omega| < |S|$), and denote the resulting function by g . Lemma 4.3 and the same calculation as in the first part of the proof imply that $g \in \overset{\circ}{H}^1(S)$ with

$$\int_S |\nabla g|^2 dx \leq \lambda \int_S |g|^2 dx.$$

As before, this implies that g is a scalar multiple of z and, after evaluation at $x = 0$, that $g = z$. This proves the assertion also in this case. \square

5. YANG INEQUALITIES FOR MAGNETIC SCHRÖDINGER OPERATORS

We turn now to the proof of Proposition 2.3. We work under the same assumptions on Ω , V and A as in the previous sections. Again we denote by u_i , $i \in \mathbb{N}$, orthonormal eigenfunctions corresponding to the eigenvalues

λ_i . Throughout the proof we will fix $k \in \mathbb{N}$. Moreover, for any $1 \leq i \leq k$ and any $1 \leq l \leq d$ we define the function

$$\varphi_i := x_l u_i - \sum_{j=1}^k a_{ij} u_j, \quad a_{ij} := \int x_l u_i \overline{u_j} dx,$$

and note that

$$(5.1) \quad (\varphi_i, u_j) = 0, \quad 1 \leq j \leq k.$$

Hence the variational principle implies

$$(5.2) \quad \lambda_{k+1} \leq \frac{\int (|(D - A)\varphi_i|^2 + V|\varphi_i|^2) dx}{\int |\varphi_i|^2 dx}$$

(with the convention that the RHS is infinite if $\varphi_i \equiv 0$). Using the commutator identity

$$(5.3) \quad [(D - A)^2, x_l] = -2i(D_l - A_l)$$

and the orthogonality (5.1) one easily finds

$$\int (|(D - A)\varphi_i|^2 + V|\varphi_i|^2) dx = \lambda_i \int |\varphi_i|^2 dx + 2 \operatorname{Im} \int (D_l - A_l) u_i \overline{\varphi_i} dx,$$

which, together with (5.2), leads to

$$(5.4) \quad \lambda_{k+1} - \lambda_i \leq \frac{2 \operatorname{Im} \int (D_l - A_l) u_i \overline{\varphi_i} dx}{\int |\varphi_i|^2 dx}.$$

Next, we manipulate the numerator. We note that

$$\begin{aligned} 2 \operatorname{Im} \int (D_l - A_l) u_i \overline{\varphi_i} dx &= -2 \operatorname{Re} \int \frac{\partial u_i}{\partial x_l} \overline{x_l u_i} dx = - \int x_l \frac{\partial |u_i|^2}{\partial x_l} dx \\ &= \int |u_i|^2 dx = 1, \end{aligned}$$

and hence

$$(5.5) \quad 2 \operatorname{Im} \int (D_l - A_l) u_i \overline{\varphi_i} dx = 1 + 2 \operatorname{Re} \sum_{j=1}^k \overline{a_{ij}} b_{ij}$$

where we have set

$$b_{ij} := i \int (D_l - A_l) u_i \overline{u_j} dx.$$

With the help of (5.3) we calculate

$$2b_{ij} = - \int \left(x_l u_i \overline{(D - A)^2 u_j} - (D - A)^2 u_i \overline{x_l u_j} \right) dx = (\lambda_i - \lambda_j) a_{ij}.$$

Hence (5.5) becomes

$$(5.6) \quad 2 \operatorname{Im} \int (D_l - A_l) u_i \overline{\varphi_i} dx = 1 + \sum_{j=1}^k (\lambda_i - \lambda_j) |a_{ij}|^2$$

Now we use the Cauchy-Schwarz inequality to estimate the numerator in (5.4). Note that by (5.1)

$$\operatorname{Im} \int (D_l - A_l) u_i \overline{\varphi_i} dx = \operatorname{Im} \int \left((D_l - A_l) u_i + i \sum_{j=1}^k b_{ij} u_j \right) \overline{\varphi_i} dx$$

and hence

$$\begin{aligned} & \left(\operatorname{Im} \int (D_l - A_l) u_i \overline{\varphi_i} dx \right)^2 \\ & \leq \left(\int |\varphi_i|^2 dx \right) \left(\int \left| (D_l - A_l) u_i + i \sum_{j=1}^k b_{ij} u_j \right|^2 dx \right) \\ & = \left(\int |\varphi_i|^2 dx \right) \left(\int |(D_l - A_l) u_i|^2 dx - \sum_{j=1}^k |b_{ij}|^2 \right) \\ & = \left(\int |\varphi_i|^2 dx \right) \left(\int |(D_l - A_l) u_i|^2 dx - \frac{1}{4} \sum_{j=1}^k (\lambda_i - \lambda_j)^2 |a_{ij}|^2 \right). \end{aligned}$$

Combining this inequality with (5.4) and (5.6) we arrive at

$$\lambda_{k+1} - \lambda_i \leq \frac{4 \int |(D_l - A_l) u_i|^2 dx - \sum_{j=1}^k (\lambda_i - \lambda_j)^2 |a_{ij}|^2}{1 + \sum_{j=1}^k (\lambda_i - \lambda_j) |a_{ij}|^2}.$$

Equivalently,

$$(5.7) \quad (\lambda_{k+1} - \lambda_i) + (\lambda_{k+1} - \lambda_j)(\lambda_i - \lambda_j) \sum_{j=1}^k |a_{ij}|^2 \leq 4 \int |(D_l - A_l) u_i|^2 dx.$$

Now we write a_{ijl} instead of a_{ij} to emphasize the dependence on l , and define

$$A_{ij} := \sum_{l=1}^d |a_{ijl}|^2.$$

Summing (5.7) over l and using that $V \geq 0$ we obtain

$$d(\lambda_{k+1} - \lambda_i) + (\lambda_{k+1} - \lambda_j)(\lambda_i - \lambda_j) \sum_{j=1}^k A_{ij} \leq 4\lambda_i.$$

Note that $A_{ij} = A_{ji}$. In order to (anti-)symmetrize, we multiply by $(\lambda_{k+1} - \lambda_i)$ and sum over i . The resulting inequality is

$$d \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq 4 \sum_{i=1}^k \lambda_i (\lambda_{k+1} - \lambda_i).$$

This completes the proof of Proposition 2.3.

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RUPERT L. FRANK, DEPARTMENT OF MATHEMATICS, ROYAL INSTITUTE OF TECHNOLOGY, 100 44 STOCKHOLM, SWEDEN

E-mail address: `rupert@math.kth.se`

ARI LAPTEV, DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON, LONDON SW7 2AZ, UK & DEPARTMENT OF MATHEMATICS, ROYAL INSTITUTE OF TECHNOLOGY, 100 44 STOCKHOLM, SWEDEN

E-mail address: `a.laptev@imperial.ac.uk` & `laptev@math.kth.se`

STANISLAV MOLCHANOV, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA, CHARLOTTE, NC 28223 - 0001, USA

E-mail address: `smolchan@unc.edu`